Suggested Solutions to HW1, MMAT5000

(1) Proof: $A = \{x \in \mathbb{Q}^+ : x^2 < 2\}$.

$B = \{y \in \mathbb{Q}^+ : y^2 > 2\}$.

Suppose that A contains a largest number α, i.e.

$\alpha \in \mathbb{Q}^+$, $\alpha^2 < 2$, and for $\forall x \in A$, $x \leq \alpha \in A$.

Set $\varepsilon = \frac{1}{2}\min\left\{1, \frac{2-\alpha^2}{(\alpha+1)^2}\right\}$, it's easy to check that $\varepsilon \in \mathbb{Q}^+$, and $\alpha+\varepsilon \in \mathbb{Q}^+$.

Moreover,

$$\begin{aligned}(\alpha+\varepsilon)^2 &= \alpha^2 + 2\alpha\varepsilon + \varepsilon^2 \\ &= \alpha^2 + \varepsilon(2\alpha+\varepsilon) \\ &< \alpha^2 + \varepsilon(2\alpha+1) \\ &< \alpha^2 + \varepsilon(\alpha^2+2\alpha+1) \\ &= \alpha^2 + \varepsilon(\alpha+1)^2 \\ &< \alpha^2 + 2 - \alpha^2 \\ &= 2.\end{aligned}$$

So

$\alpha+\varepsilon \in A$ but $\alpha+\varepsilon > \alpha$, a contradiction!

Hence, A contains no largest number

Similarly, suppose that B contains a smallest number β, i.e.

$\beta \in \mathbb{Q}^+$, $\beta^2 > 2$ and for $\forall y \in B$, $y \geq \beta$.

~~Set $\eta = \frac{\beta^2-2}{2(1+\beta)^2}$, it's easy to check that $\eta \in \mathbb{Q}^+$~~

Set $\eta = \frac{1}{2}\min\left\{\beta, \frac{\beta^2-2}{(1+\beta)^2}\right\}$, it's easy to check that $\beta-\eta \in \mathbb{Q}^+$, $\eta \in \mathbb{Q}^+$.

Moreover,

$$\begin{aligned}(\beta-\eta)^2 &= \beta^2 - 2\beta\eta + \eta^2 = \beta^2 - \eta(2\beta-\eta) \\ &> \beta^2 - \eta(2\beta+1+\beta^2) \\ &= \beta^2 - \eta(\beta+1)^2 > \beta^2 - (\beta^2-2) = 2.\end{aligned}$$

So

$$\beta-\eta \in B \quad \text{but} \quad \beta-\eta < \beta \ , \quad \text{a contradiction!}$$

Hence, B contains no smallest number.

(2). Solution:

a) $\sup S_1 = \max\{\sup A, \sup B\}$

b) $\sup S_2 \le \min\{\sup A, \sup B\}$.

c) $\sup S_3 = \sup A + \sup B$

d) $\inf S_3 = \inf A + \inf B$

e) $\sup S_4 = \sup A + a$. $\quad (a>0)$

f) $\sup S_5 = a \cdot \sup A$. $\quad (a>0)$

(3) Proof: (i) Let $\delta_n = n^{\frac{1}{n}} - 1$.

It suffices to show that $\lim\limits_{n\to\infty} \delta_n = 0$.

Notice that

$$n = (1+\delta_n)^n \geqslant 1 + \frac{n(n-1)}{2}\delta_n^2$$

we have

$$\delta_n \le \sqrt{\frac{2}{n}}$$

So for any given $\varepsilon > 0$, we need only

$$\sqrt{\frac{2}{n}} < \varepsilon \ ,$$

which is equivalent to $n > \frac{2}{\varepsilon^2}$.

Thus, let $N \triangleq \left[\frac{2}{\varepsilon^2}\right] + 1$. (Here $[f]$ denotes the integer part of f).

Then for all $n > N$,

$$|\delta_n| < \varepsilon \ ,$$

which shows that $\lim\limits_{n\to\infty} \delta_n = 0$ i.e. $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$.

If $\varepsilon = 0.1$, then our $N = 201$.

(ii) For any given $\varepsilon > 0$, there exists $N = [\frac{1}{\varepsilon}] + 1 > 0$. Such that for all $n > N$,

$$\left| \frac{n!}{n^n} - 0 \right| = \frac{1 \times 2 \times \cdots \times (n-1) \times n}{n \times n \times \cdots \times n \times n} \leq \frac{1}{n} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

If $\varepsilon = 0.1$, then $N = 11$.

(4) <u>Solution</u>:

(i) <u>Case ①</u>: $c = 2$,

We have $a_n = 2$ for all $n \geq 1$.

So $\lim_{n\to\infty} a_n = 2$.

<u>Case ②</u>: $0 < c < 2$.

we have $0 < a_n < 2$ (by induction) for all $n \geq 1$.

Moreover,

$$\begin{aligned} a_{n+1} - a_n &= \sqrt{2a_n} - a_n \\ &= \sqrt{a_n}(\sqrt{2} - \sqrt{a_n}) \\ &> 0. \end{aligned}$$

So $\{a_n\}$ is an increasing sequence with upper bound, and hence converges. Set $l = \lim_{n\to\infty} a_n$, then

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{2a_n} \\ l &= \sqrt{2l}. \\ l &= 2. \end{aligned}$$

(Here we rejected the solution $l = 0$ since $a_n \geq a_1 = c$)

Case 3: $c > 2$.

We have $a_n > 2$ (by induction) for all $n \geq 1$.

Moreover,

$$a_{n+1} - a_n = \sqrt{2a_n} - a_n$$
$$= \sqrt{a_n}\,(\sqrt{2} - \sqrt{a_n})$$
$$< 0$$

So $\{a_n\}$ is a decreasing sequence with lower bound, and hence converges.

Set $l = \lim_{n\to\infty} a_n$, then

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n}$$
$$l = \sqrt{2l}$$
$$l = 2 \quad \text{(Here we rejected the solution } l = 0 \text{ since } a_n > 2 \text{ for all } n \geq 1\text{).}$$

In all, $\lim_{n\to\infty} a_n = 2$.

Next, we show that $\{a_n\}$ is a Cauchy sequence.

Since $\{a_n\}$ converges to 2, for any given $\varepsilon > 0$, there exists $N > 0$. Such that if $j, k > N$,

$$|a_j - 2| \leq \varepsilon/2.$$
$$|a_k - 2| < \varepsilon/2.$$

and hence

$$|a_j - a_k| = |a_j - 2 + 2 - a_k|$$
$$\leq |a_j - 2| + |a_k - 2|$$
$$\leq \varepsilon \qquad \text{for all } j, k > N.$$

which implies that $\{a_n\}$ is a Cauchy sequence.

(ii)

① when $C=2$, then we have $a_n = 2$ for all $n \geq 1$,

So $\lim_{n\to\infty} a_n = 2$.

② when $0<C<2$, then we have $0<a_n<2$ for all $n \geq 1$.

Moreover, for all $n \geq 1$,

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n$$

$$= \frac{2+a_n - a_n^2}{\sqrt{2+a_n} + a_n}$$

$$= \frac{(1+a_n)(2-a_n)}{\sqrt{2+a_n} + a_n} > 0.$$

So $\{a_n\}$ is an increasing sequence with upper bound, and hence converges.

Set $l = \lim_{n\to\infty} a_n$, then we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$$

$$l = \sqrt{2+l}$$

$$l = 2$$

③ When $c>2$, then we have that $a_n > 2$ for all $n \geq 1$.

Moreover, for all $n \geq 1$,

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n$$

$$= \frac{(1+a_n)(2-a_n)}{\sqrt{2+a_n} + a_n} < 0,$$

which implies that $\{a_n\}$ is an decreasing sequence.

since $a_n > 2$ for all $n \geq 1$, we have that $\{a_n\}$ converges.

Set $l = \lim_{n\to\infty} a_n$, then we have

$$l = \sqrt{2+l}$$

$$l = 2.$$

In all, $\lim_{n\to\infty} a_n = 2$.

$\{a_n\}$ is a ~~th~~ Cauchy sequence, since we could show it similarly as (i).

(iii) Since $a_1 = c > 0$, it's easy to check that $a_n > 0$ for all $n \geq 1$.

Moreover

$$a_{n+1} - \sqrt{2} = 1 + \frac{1}{1+a_n} - \sqrt{2}$$

$$= \frac{1-\sqrt{2}}{1+a_n}(a_n - \sqrt{2})$$

Notice that

$$\left|\frac{1-\sqrt{2}}{1+a_n}\right| \leq \sqrt{2} - 1 < \frac{1}{2}$$

So for any $n \geq 1$, we have

$$\begin{aligned} |a_{n+1} - \sqrt{2}| &< \frac{1}{2}|a_n - \sqrt{2}| \\ &< \frac{1}{2^2}|a_{n-1} - \sqrt{2}| \\ &< \cdots \\ &< \frac{1}{2^n}|a_1 - \sqrt{2}| \\ &= \frac{1}{2^n}|c - \sqrt{2}| \end{aligned}$$

So

$$\lim_{n\to\infty}|a_n - \sqrt{2}| = \lim_{n\to\infty}\frac{1}{2^{n-1}} \cdot |c - \sqrt{2}| = 0$$

i.e.

$$\lim_{n\to\infty} a_n = \sqrt{2}.$$

Since $\{a_n\}$ converges, it is Cauchy.

Remark: In (iii), $\{a_n\}$ is NOT monotonic when $c \neq \sqrt{2}$, since we have.

$$(a_{n+1} - \sqrt{2})(a_n - \sqrt{2}) \cancel{\frac{1-\sqrt{2}}{1+a_n}} < 0 \quad \left(\text{when } c \neq \sqrt{2}\right)$$